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A unified approach to infinitesimal Loewner and Geroch transformations and the Ernst and Einstein–Maxwell equations

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Abstract
It is demonstrated that the dual Ernst equation of general relativity constitutes a stationary Loewner system. In an analogous manner, it is shown that the Einstein–Maxwell equations for stationary axisymmetric space-times and their extension to Einstein's equations coupled with an arbitrary number of $U(1)$ gauge fields may be interpreted as generalized Loewner systems. Moreover, it is recorded that the base Geroch transformation for the (dual) Ernst equation may be located in a class of infinitesimal Bäcklund transformations introduced by Loewner in 1952. A Geroch-type transformation for a generic class of 2 + 1-dimensional Loewner systems is set down and it is shown how the base Geroch and Hoenselaers–Kinnersley–Xanthopoulos transformations are naturally retrieved.

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1. Introduction

The integrable nature of Einstein's vacuum equations for stationary axisymmetric space-times was indicated and established around 1978 by a variety of individuals and groups such as Belinsky and Zakharov [2], Harrison [9], Kinnersley and Chitre [12–14], Maison [23] and Neugebauer [26]. There has since then been extensive research on the application of solution generation techniques to the Ernst and Einstein–Maxwell equations and other integrable cases of Einstein's equations (see [11] and references therein). However, in 1972, albeit at a somewhat theoretical level, it was shown by Geroch [7] that, in principle, an internal symmetry group may be exploited to generate stationary axisymmetric space-times in vacuum of arbitrary complexity. Later, it was indeed established that other solution generation techniques may be traced back to this 'Geroch group'. An account of the connections between group-theoretic and soliton-theoretic methods for generating solutions of Einstein's equations has been given by Cosgrove in [3].

In 1950, Loewner [21] introduced a generalization of the notion of classical Bäcklund transformations. His investigations were concerned with a search for multi-parameter pressure–density laws for which the hodograph equations of plane gasdynamics may be reduced by finite Bäcklund transformations to tractable forms in various flow régimes. Only recently [38], these have been identified as binary Darboux-type transformations which are standard in soliton theory [24]. In 1952, Loewner [22] introduced in the same context the concept of infinitesimal Bäcklund transformations. It was not until 1991 that, suitably reinterpreted and extended, Loewner's infinitesimal Bäcklund transformations were also shown to have profound connections with soliton theory. Thus, Konopelchenko and Rogers [15, 16] proposed regarding the continuous parameter in Loewner's infinitesimal Bäcklund transformation as a third independent variable so that Loewner's consistency conditions may be interpreted as a 2 + 1-dimensional system of nonlinear matrix differential equations. This approach led to the discovery of a long-sought integrable 2 + 1-dimensional generalization of the classical sine-Gordon equation wherein the two 'spatial' variables occur on an equal footing. Since its introduction, the ubiquitous character of the integrable Loewner (–Konopelchenko–Rogers) system in mathematics and mathematical physics, especially in the context of classical differential geometry, has been well documented [27–29, 31–37, 39].

In this paper, infinitesimal Geroch and Loewner transformations are brought together for the first time. Specifically, we show that the base Geroch transformation for the dual Ernst equation which may be regarded as the 'master equation' for stationary axisymmetric fields constitutes nothing but a particular Loewner transformation. This result was instigated by the earlier observation that the Ernst equation and, more generally, the Ernst–Weyl equation descriptive of the interaction of neutrino and gravitational fields [1, 41] represent canonical $2+0$-dimensional reductions of the Loewner system [31, 33]. In fact, we here demonstrate that the stationary axisymmetric Einstein–Maxwell field equations and their extension to Einstein's equations coupled with an arbitrary number of $U(1)$ gauge fields may likewise be interpreted as stationary generalized Loewner systems. Generalized Loewner systems have been introduced in [29] and constitute 'squared eigenfunction' symmetries of the multi-component (modified) Kadomtsev–Petviashvili ((m)KP) hierarchy.

The Geroch–Loewner connection is exploited to reformulate the base Geroch transformation for the dual Ernst equation in a manner which allows immediate generalization. Thus, we here present an infinitesimal Geroch-type transformation for the 2 + 1-dimensional Loewner system which reduces to the standard base Geroch transformation in the case of Einstein's equations. By construction, this Geroch-type transformation encapsulates another 2 + 1-dimensional Loewner system which is compatible with the original Loewner system. In fact, it is readily seen that, in the generic case, the two Loewner systems appear in a symmetric manner so that either Loewner system may be regarded as defining an infinitesimal Geroch-type transformation for the other Loewner system. This observation sheds new light on the relation between the dual Ernst equation and its base Geroch transformation. However, it is shown that in a degenerate case, the infinitesimal Geroch-type transformation may be integrated explicitly to obtain a particular case of Loewner's finite Bäcklund transformations. Remarkably, in the context of the dual Ernst equation, the base Hoenselaers–Kinnersley–Xanthopoulos (HKX) transformation [10] is retrieved.

The results presented in this paper may now be used to study the action of infinitesimal and finite Geroch-type transformations on any integrable system which resides in the Loewner system. Since the Loewner system has been shown to encode a variety of geometrically significant systems, a geometric interpretation of the Geroch-type transformation is of particular interest. Moreover, the application of Geroch-type transformations not only in Loewner's original gasdynamics setting but also in nonlinear elastodynamics and electromagnetic wave propagation [30] may now be investigated.

2. The Ernst equation and its base Geroch transformation

It is well known that the metric of a four-dimensional space-time with two commuting space- and time-like Killing vectors may be cast into block-diagonal form

$$\mathrm{d}s^2 = \mathrm{e}^{2k}(\mathrm{d}x^2 + \mathrm{d}y^2) + g_{33}\,\mathrm{d}\varphi^2 + 2g_{34}\,\mathrm{d}\varphi\,\mathrm{d}t + g_{44}\,\mathrm{d}t^2 \tag{1}$$

if the Killing vectors are orthogonal transitive. Such space-times are termed stationary and axisymmetric. Here, the metric coefficients depend on x and y only. In terms of the trace-free matrix-valued function

$$F = \begin{pmatrix} g_{34} & g_{33} \\ -g_{44} & -g_{34} \end{pmatrix}, \tag{2}$$

Einstein's vacuum field equations

$$R_{\mu\nu} = 0 \tag{3}$$

then reduce to [12]

$$\nabla \cdot \left(\frac{1}{\rho} F \nabla F\right) \qquad \Delta\rho = 0 \tag{4}$$

with the definitions

$$\rho^2 = -\det F \qquad \nabla = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} \qquad \Delta = \nabla^2. \tag{5}$$

The remaining metric coefficient k is obtained in terms of quadratures.

2.1. The Ernst equation and its dual

A canonical parametrization of the matrix F is given by

$$F = \frac{\rho}{\mathcal{F} + \mathcal{F}^*} \begin{pmatrix} \mathcal{F} - \mathcal{F}^* & 2\mathcal{F}\mathcal{F}^* \\ 2 & \mathcal{F}^* - \mathcal{F} \end{pmatrix} \tag{6}$$

so that the field equations reduce to

$$\Delta\mathcal{F} + \frac{\nabla\rho \cdot \nabla\mathcal{F}}{\rho} = 2\frac{(\nabla\mathcal{F})^2}{\mathcal{F} + \mathcal{F}^*} \tag{7}$$

and its 'conjugate' obtained by formally applying the star operations $(\mathcal{F})^* = \mathcal{F}^*$ and $(\mathcal{F}^*)^* = \mathcal{F}$ together with the harmonicity condition $\Delta\rho = 0$. Alternatively, one may exploit the 'conservative' form of $(4)_1$ and introduce a matrix-valued potential Ω according to

$$\tilde{\nabla}\Omega = -\frac{1}{\rho} F \nabla F \qquad \tilde{\nabla} = \begin{pmatrix} \partial_y \\ -\partial_x \end{pmatrix}. \tag{8}$$

It is noted that $\nabla \cdot \tilde{\nabla} = 0$. If we now set

$$H = F + \mathrm{i}\Omega \tag{9}$$

then the important linear relation

$$\nabla H = \frac{\mathrm{i}}{\rho} F \tilde{\nabla} H \tag{10}$$

is obtained. In fact, by construction, the field equations are equivalent to (10) subject to the constraint

$$\Re(H) = F. \tag{11}$$

The latter is the origin of the nonlinear character of the field equations.

The complex coefficient

$$\mathcal{E} = H_{21} \tag{12}$$

is readily shown to obey the Ernst equation [6]

$$\Delta\mathcal{E} + \frac{\nabla\rho\cdot\nabla\mathcal{E}}{\rho} = \frac{(\nabla\mathcal{E})^2}{\Re(\mathcal{E})}. \tag{13}$$

Since the Ernst equation and (7) are formally related by the Kramer–Neugebauer transformation [19]

$$(\mathfrak{S}) \qquad (\mathcal{F}, \mathcal{F}^*) \to (\mathcal{E}, \bar{\mathcal{E}}), \tag{14}$$

the latter equation is regarded as dual to the Ernst equation. Specifically, if, for convenience, we adopt the Lewis–Papapetrou form [20]

$$\mathrm{d}s^2 = \frac{1}{f}(\mathrm{e}^{2\gamma}(\mathrm{d}x^2 + \mathrm{d}y^2) + \rho^2\,\mathrm{d}\varphi^2) - f(\mathrm{d}t - \omega\,\mathrm{d}\varphi)^2 \tag{15}$$

of the space-time metric then the Ernst potential

$$\mathcal{E} = f + \mathrm{i}\psi \tag{16}$$

is related to the metric coefficients by the contact transformation encoded in the lower-left component of (8), that is

$$\tilde{\nabla}\omega = \frac{\rho}{f^2}\nabla\psi. \tag{17}$$

Conversely, for any solution $\mathcal{E}$ of the Ernst equation, the matrix

$$F = \begin{pmatrix} f\omega & \rho^2/f - f\omega^2 \\ f & -f\omega \end{pmatrix}, \tag{18}$$

where the functions f and ω are defined by (16) and (17) respectively, may be shown to obey (4).

2.2. *The base Geroch transformation*

It is evident that the matrix equation (4) is invariant under the Matzner–Misner transformation [25]

$$(\mathfrak{M}) \qquad F \to C^{-1}FC, \tag{19}$$

where C is a nonsingular constant matrix. By construction, this transformation acts directly on the metric coefficients g_{ik} and may be compensated for by a linear transformation of the ignorable coordinates φ and t. The Matzner–Misner transformation may be interpreted as a Möbius transformation of the form

$$\mathcal{F} \to \frac{a\mathcal{F} + b}{c\mathcal{F} + d} \qquad \mathcal{F}^* \to \frac{a\mathcal{F}^* - b}{-c\mathcal{F}^* + d} \tag{20}$$

acting on the dual Ernst equation (7). Here, the constants a, b, c and d are defined in terms of the matrix C. The analogue of the Möbius transformation (20) for the Ernst equation is given by the Ehlers transformation [5]

$$(\mathfrak{E}) \qquad \mathcal{E} \to \frac{\tilde{a}\mathcal{E} + \mathrm{i}\tilde{b}}{\mathrm{i}\tilde{c}\mathcal{E} + \tilde{d}}, \tag{21}$$

where the constants $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ and $\tilde{d}$ are real.

The transformations $\mathfrak{S}$, $\mathfrak{M}$ and $\mathfrak{E}$ may be exploited in the following manner. Given a solution $(\mathcal{F}, \mathcal{F}^*)$ of the dual Ernst equation (or, equivalently, a space-time metric), one first applies the Matzner–Misner transformation $\mathfrak{M}$ and then maps the new solution of the dual Ernst equation to a solution of the Ernst equation by means of the Kramer–Neugebauer transformation $\mathfrak{S}$. The Ehlers transformation $\mathfrak{E}$ is then applied and the result is mapped to another solution of the dual Ernst equation via the inverse of the Neugebauer transformation $\mathfrak{S}^{-1}$. The composite transformation

$$(\mathfrak{G}) \qquad (\mathcal{F}, \mathcal{F}^*) \to (\mathfrak{S}^{-1} \circ \mathfrak{E} \circ \mathfrak{S} \circ \mathfrak{M})(\mathcal{F}, \mathcal{F}^*) \tag{22}$$

is known as a Geroch transformation [7]. The infinitesimal action of the base Geroch transformation $\mathfrak{G}$ on F has been shown to be

$$F_\epsilon = [\Omega\gamma, F] \tag{23}$$

where γ is a trace-free but otherwise arbitrary constant matrix.

In principle, it is possible to generate solutions of the stationary axisymmetric vacuum gravitational field equations which contain an arbitrary number of parameters by successive application of the base Geroch transformation (22). This is due to the fact that base Geroch transformations with different sets of parameters generally do not commute. However, in practice, the explicit representation of Geroch transformations has proven to be highly nontrivial and requires the introduction of an infinite hierarchy of so-called Kinnersley–Chitre potentials [13, 14]. These potentials have been used to show that the collection of Geroch transformations forms an infinite-dimensional Banach Lie group [40]. An important subset of this group constitute the finite HKX transformations [10]. The fundamental importance of the Geroch group in connection with other solution generation techniques for Einstein's equation has been discussed in detail in [3].

3. The Loewner connection

3.1. A class of Loewner systems

In 1952, Loewner [22], in the context of reduction of the hodograph system of plane gasdynamics to canonical form, introduced the notion of infinitesimal Bäcklund transformations for the linear matrix equation

$$\phi_y = S\phi_x. \tag{24}$$

In particular, he sought infinitesimal transformations of the form

$$\phi_{xt} = V\phi_x + W\phi \qquad \phi_{yt} = \tilde{V}\phi_y + \tilde{W}\phi \tag{25}$$

which leave the hodograph-type system (24) invariant. Here, all matrices depend parametrically on t, that is $S = S(x, y; t)$ etc. In 1991, Konopelchenko and Rogers [15] reinterpreted these Bäcklund transformations in a soliton-theoretical setting in terms of 2 + 1-dimensional integrable systems. Thus, if the parameter t is regarded as a third independent variable then the triad (24), (25) may be viewed as a linear representation of the nonlinear matrix system obtained from its compatibility conditions, namely

$$S_t = [V, S] \qquad V_y - V_x S + [W, S] = 0 \qquad W_y = (SW)_x \tag{26}$$

together with $\tilde{V} = V$ and $\tilde{W} = SW$. In this interpretation, the Loewner system has been shown to lead to a wide class of solitonic equations of physical or mathematical interest either by way of inclusion or compatibility [16,29,33,36,37,39]. For instance, a 2 + 1-dimensional integrable extension of the classical sine-Gordon equation has been obtained by Konopelchenko and Rogers [15–17] via the canonical constraints $S^2 = \mathbb{1}$, $\mathrm{tr}\, S = 0$.

A natural eigenfunction parametrization of the Loewner system is obtained by introducing a matrix potential Φ according to

$$\Phi_x = W \qquad \Phi_y = SW \tag{27}$$

and a matrix-valued function ψ via

$$V = \Phi + \psi^\dagger \tag{28}$$

where † denotes Hermitian conjugation. The Loewner system (26) then reads

$$S_t = [\Phi + \psi^\dagger, S] \qquad \Phi_y = S\Phi_x \qquad \psi_y = S^\dagger \psi_x. \tag{29}$$

Thus, Φ may be regarded as an 'eigenfunction' in that it is another solution of the linear equation (24). However, it is emphasized that, in general, Φ does *not* obey the remaining pair (25). The quantity ψ constitutes an adjoint eigenfunction since $(29)_3$ is nothing but the 'integrated' adjoint of (24). Furthermore, the Loewner system in the form (29) gives rise to a bilinear potential $M = M(\psi, \phi)$ defined by

$$M_x = \psi^\dagger \phi_x \qquad M_y = \psi^\dagger \phi_y. \tag{30}$$

The pair (25) may therefore be 'integrated' to obtain a 'nonlocal' linear representation of the Loewner system, namely

$$\phi_y = S\phi_x \qquad \phi_t = \Phi\phi + M(\psi, \phi). \tag{31}$$

It is directly verified that the above linear system is indeed compatible modulo the Loewner system (29).

3.2. The dual Ernst equation

In [31], it has been shown that the Ernst equation (13) constitutes a particular stationary Loewner system. By virtue of the Kramer–Neugebauer transformation (14), the dual Ernst equation should also be a 2 + 0-dimensional reduction of the Loewner system. Indeed, if one sets

$$S = -\frac{\mathrm{i}}{\rho} F \qquad V = 2(F + \mathrm{i}\nu \mathbb{1}) \qquad W = H_x \tag{32}$$

where ν is conjugate to the harmonic function ρ, that is

$$\nu_x = \rho_y \qquad \nu_y = -\rho_x \tag{33}$$

then the field equations (4) or, equivalently, the dual Ernst equation (7) may be cast into the form

$$[V, S] = 0 \qquad V_y - V_x S + [W, S] = 0 \qquad W_y = (SW)_x. \tag{34}$$

Thus, a stationary Loewner system is obtained wherein, remarkably, the matrix S encapsulates the nontrivial part of the space-time metric (1). We observe in passing that, if one ignores the implicit constraint $\Re(H) = F$ then the system (34) constitutes a slight generalization of the dual Ernst equation. At the level of the Ernst potential $\mathcal{E}$, it has been shown [33] that this corresponds to a generalized Ernst equation which governs neutrino and gravitational fields in axially symmetric space-times [1, 41].

In view of the following, it proves instructive to interpret the above result in terms of the eigenfunction parametrization (29) of the Loewner system. To this end, we first observe that the matrix F admits the discrete symmetry

$$F^\intercal = \sigma F \sigma \qquad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{35}$$

since F is trace-free. The quantity $\hat{H}$ defined by

$$\hat{H} = \sigma H \sigma \tag{36}$$

therefore obeys the linear equation

$$\nabla \hat{H} = -\frac{\mathrm{i}}{\rho} F^{\mathrm{T}} \tilde{\nabla} \hat{H} \tag{37}$$

which represents the adjoint of the key linear equation (10). Thus, H and $\hat{H}$ constitute (adjoint) 'eigenfunctions'. Moreover, the definition (8) of Ω implies that

$$\tilde{\nabla} \Omega^{\mathrm{T}} = \sigma \tilde{\nabla} \Omega \sigma + 2 \tilde{\nabla} \nu \, \mathbb{I}. \tag{38}$$

Without loss of generality, we may assume that

$$\Omega^{\mathrm{T}} = \sigma \Omega \sigma + 2\nu \, \mathbb{I} \tag{39}$$

which results in the identity

$$\hat{H}^{\dagger} = -H + 2(F + \mathrm{i}\nu \, \mathbb{I}). \tag{40}$$

If we now set

$$S = -\frac{\mathrm{i}}{\rho} F \qquad \psi = \hat{H} \qquad \Phi = H \tag{41}$$

then we obtain

$$[\Phi + \psi^{\dagger}, S] = 0 \qquad \Phi_y = S \Phi_x \quad \psi_y = S^{\dagger} \psi_x. \tag{42}$$

It is evident that the representations (32) and (41) are equivalent. In conclusion, it is remarked that relations of the type (36) between eigenfunctions and adjoint eigenfunctions are associated with standard admissible reductions of the Loewner system [29].

3.3. The base Geroch transformation

In the preceding, we have identified the linear equations (10) and $(29)_2$ in order to establish the Loewner–Ernst connection. Since both Loewner and Geroch provide invariances of this linear equation and (23) and $(26)_1$ are identical in form, there may exist a connection between Loewner's infinitesimal Bäcklund transformations and the infinitesimal Geroch transformations. This proves to be the case. Thus, if we set

$$V' = \Omega \gamma + \frac{1}{2\mathrm{i}} \operatorname{tr}(\bar{H} \gamma) \, \mathbb{I} \qquad W' = \frac{1}{2\mathrm{i}} H_x \gamma \tag{43}$$

then it is readily seen that

$$S_{\epsilon} = [V', S] \qquad V'_y - V'_x S + [W', S] = 0 \qquad W'_y = (SW')_x. \tag{44}$$

Accordingly, we obtain the remarkable result that the base Geroch transformation may be interpreted as a 2 + 1-dimensional Loewner system and, in turn, a particular infinitesimal Loewner transformation. In fact, in order to verify this result, it is sufficient to show that the quantity ψ' defined by

$$V' = \Phi' + \psi'^{\dagger} \qquad \Phi' = \frac{1}{2\mathrm{i}} H \gamma \tag{45}$$

constitutes an adjoint eigenfunction. To this end, it is noted that $(45)_1$ may be written as

$$\psi'^{\dagger} = \bar{\Phi}' - \operatorname{tr}(\bar{\Phi}') \, \mathbb{I} \tag{46}$$

which simplifies to

$$\psi' = \sigma \Phi' \sigma \tag{47}$$

by virtue of the identity

$$Q^{\top} = \sigma Q \sigma + \mathrm{tr}(Q)\,\mathbb{1} \tag{48}$$

for any 2×2 matrix Q and hence ψ' is indeed an adjoint eigenfunction. It is also noted that the (adjoint) eigenfunctions Φ' and ψ' associated with the base Geroch transformation are related to those corresponding to the dual Ernst equation by

$$\Phi' = \Phi\tau \qquad \psi' = \psi\tau^{\dagger} \qquad \tau = \frac{1}{2\mathrm{i}}\sigma \tag{49}$$

whence

$$S_{\epsilon} = [\Phi\tau + \tau\psi^{\dagger}, S]. \tag{50}$$

3.4. A Geroch-type transformation for the generic Loewner system

The above observation may be exploited to define Geroch-type transformations for any 2 + 1-dimensional Loewner system. Indeed, it is readily verified that the Loewner system (29) is invariant under the evolution

$$\begin{aligned} S_{\epsilon} &= [\Phi\tau + \tau\psi^{\dagger}, S] \\ \Phi_{\epsilon} &= [\tau, N] + [\Phi\tau, \Phi] + \Phi_t\tau \\ \psi^{\dagger}_{\epsilon} &= [\tau, \psi^{\dagger}\Phi - N] + [\tau\psi^{\dagger}, \psi^{\dagger}] + \tau\psi^{\dagger}_t, \end{aligned} \tag{51}$$

where the bilinear potential N is defined by the compatible equations

$$N_x = \psi^{\dagger}\Phi_x \qquad N_y = \psi^{\dagger}\Phi_y \tag{52}$$

and τ denotes an arbitrary constant matrix. It is noted that the quantities

$$S \qquad \Phi' = \Phi\tau \qquad \psi' = \psi\tau^{\dagger} \tag{53}$$

constitute a solution of another 2 + 1-dimensional Loewner system, namely

$$S_{\epsilon} = [\Phi' + \psi'^{\dagger}, S] \qquad \Phi'_y = S\Phi'_x \quad \psi'_y = S^{\dagger}\psi'_x. \tag{54}$$

Thus, it has been established that this Loewner system is *compatible* with the original Loewner system (29). By construction, in the case of the dual Ernst equation, the Loewner system (54) reduces to that associated with the base Geroch transformation.

3.5. HKX transformations

As mentioned in the previous section, infinitesimal Geroch transformations may be integrated explicitly only under particular circumstances. For instance, the class of finite HKX transformations [10] is obtained by choosing a degenerate matrix γ, that is

$$\gamma^2 = 0 \tag{55}$$

without loss of generality. This constraint arises naturally in the context of the Geroch-type transformation (51) for the Loewner system discussed in the preceding. Thus, the ϵ-evolutions $(51)_{2,3}$ for Φ and ψ imply that

$$\begin{aligned} (\Phi'_{\epsilon} - \Phi'^2 - N') &= (\Phi_t - \Phi^2 - N)\tau^2 \\ (\psi'^{\dagger}_{\epsilon} + \psi'^{\dagger 2} + \psi'^{\dagger}\Phi' - N') &= \tau^2(\psi^{\dagger}_t + \psi^{\dagger 2} + \psi^{\dagger}\Phi - N). \end{aligned} \tag{56}$$

If the matrix τ is nonsingular then it is concluded that the Loewner systems (29) and (54) occur on an equal footing. This implies that the original Loewner system (29) likewise encapsulates an infinitesimal Geroch-type transformation for the Loewner system (54). However, if we impose the constraint

$$\tau^2 = 0 \tag{57}$$

then this symmetry is broken and Φ' and ψ' must obey the ϵ-evolutions

$$\Phi'_\epsilon = \Phi'^2 + N' \qquad \psi'^\dagger_\epsilon = -\psi'^{\dagger 2} - \psi'^\dagger \Phi' + N'. \tag{58}$$

These are indeed compatible with the Loewner system (54). This is evident since comparison with the nonlocal Lax pair

$$\phi'_y = S\phi'_x \qquad \phi'_\epsilon = \Phi'\phi' + M(\psi', \phi') \tag{59}$$

for the Loewner system and its 'adjoint' [29] reveals that, for instance, the condition $(58)_1$ is equivalent to demanding that Φ' be a 'proper' eigenfunction which satisfies both $(59)_1$ *and* $(59)_2$.

Remarkably, it has been shown [38] that if one assumes that the matrix eigenfunction Φ associated with any 2+1-dimensional Loewner system (26) also satisfies the 't-evolutions' (25) and the adjoint eigenfunction ψ obeys the adjoint evolutions then the Loewner system may be integrated and one obtains Loewner's finite Bäcklund transformations [21] introduced in 1950 which, in turn, have been identified as standard binary Darboux transformations [24]. Thus, the base HKX transformation constitutes a particular finite Loewner transformation. This is in agreement with the fact that HKX transformations may be related to Harrison transformations [9] in certain confluence limits [4].

4. Generalized Loewner systems and Einstein–Maxwell equations

4.1. Stationary axisymmetric Einstein–Maxwell field equations

In 1977, Kinnersley showed that the Einstein–Maxwell equations

$$\begin{aligned} G_{\mu\nu} &= \mathsf{F}_{\mu\alpha}\mathsf{F}_\nu{}^\alpha - \tfrac{1}{4} g_{\mu\nu}\mathsf{F}_{\alpha\beta}\mathsf{F}^{\alpha\beta} \\ \mathsf{F}^{\mu\nu}{}_{;\nu} &= 0 \qquad \mathsf{F}_{\mu\nu} = \mathsf{A}_{\nu,\mu} - \mathsf{A}_{\mu,\nu} \end{aligned} \tag{60}$$

where $G_{\mu\nu}, \mathsf{F}_{\mu\nu}$ and A_μ denote the Einstein and electromagnetic field tensors and the electromagnetic four-vector potential, respectively, may be cast into a '*most attractive form*' [12] in the case of axisymmetric stationary space-times. Thus, in terms of a complex matrix H, a complex vector φ and the matrix F introduced in the preceding, the field equations may be formulated as

$$\nabla H = \frac{\mathrm{i}}{\rho} F\tilde{\nabla} H \qquad \nabla\varphi = \frac{\mathrm{i}}{\rho} F\tilde{\nabla}\varphi \tag{61}$$

subject to the constraint

$$\Re(H) = F + \Re(\varphi\varphi^\dagger\sigma + \kappa\,\mathbb{I}), \tag{62}$$

where the 'quadratic' potential κ is defined by

$$\nabla\kappa = \varphi^\dagger\sigma\nabla\varphi. \tag{63}$$

Here, the key observation is that H and φ satisfy the *same linear* equation. Once again, the nonlinearity of the field equations is due to the constraint (62).

Decomposition of H and φ into

$$H = F + \Re(\varphi\varphi^\dagger\sigma) + \mathrm{i}[\Xi - \Im(\varphi\varphi^\intercal\sigma)] + \kappa\,\mathbb{I} \qquad \varphi = A + \mathrm{i}B \tag{64}$$

yields

$$\tilde{\nabla}\Xi = -\frac{1}{\rho}[F\nabla F + 2F(\nabla A)A^{\mathsf{T}}\sigma - 2A(\nabla A)^{\mathsf{T}}\sigma F]$$
$$\tilde{\nabla}B = -\frac{1}{\rho}F\nabla A. \tag{65}$$

The compatibility conditions for $(65)_1$ and $(65)_2$ then result in the nontrivial parts of Einstein's equations $(60)_1$ and Maxwell's equations $(60)_{2,3}$ respectively:

$$\nabla\cdot\left(\frac{1}{\rho}[F\nabla F + 2F(\nabla A)A^{\mathsf{T}}\sigma - 2A(\nabla A)^{\mathsf{T}}\sigma F]\right) = 0$$
$$\nabla\cdot\left(\frac{1}{\rho}F\nabla A\right) = 0. \tag{66}$$

The Ernst potential $\mathcal{E}$ and the electromagnetic potential Φ may now be defined as $\mathcal{E} = H_{21}$ and $\Phi = \varphi_2$, in terms of which the field equations read [6]

$$\frac{f}{\rho}\nabla\cdot(\rho\nabla\mathcal{E}) = (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\mathcal{E}$$
$$\frac{f}{\rho}\nabla\cdot(\rho\nabla\Phi) = (\nabla\mathcal{E} + 2\bar{\Phi}\nabla\Phi)\cdot\nabla\Phi \tag{67}$$

with $f = \Re(\mathcal{E}) + |\Phi|^2$.

4.2. Generalized Loewner systems

Generalized Loewner systems were introduced in [29] in connection with squared eigenfunction symmetries of the multi-component (m)KP hierarchy. Here, we focus on a particular subclass of generalized Loewner systems. Thus, a natural extension of the linear representation (31) for the Loewner system in the form (29) is obtained by introducing a finite number of (adjoint) eigenfunctions Φ_n, ψ_n obeying the linear equations

$$\Phi_{ny} = S\Phi_{nx} \qquad \psi_{ny} = S^{\dagger}\psi_{nx} \tag{68}$$

and associated bilinear potentials $M_n = M(\psi_n, \phi)$ defined by

$$M_{nx} = \psi_n^{\dagger}\phi_x \qquad M_{ny} = \psi_n^{\dagger}\phi_y. \tag{69}$$

Here, any pair (Φ_n, ψ_n) may be either matrix-valued or vector-valued. It is now readily verified that the linear system

$$\phi_y = S\phi_x \qquad \phi_t = \sum_n \Phi_n M(\psi_n, \phi) \tag{70}$$

is compatible modulo

$$S_t = \left[\sum_n \Phi_n\psi_n^{\dagger}, S\right] \qquad \Phi_{ny} = S\Phi_{nx} \quad \psi_{ny} = S^{\dagger}\psi_{nx}. \tag{71}$$

The generalized Loewner system (71) reduces to the standard Loewner system (29) if one makes the choice

$$\Phi_1 = \Phi \qquad \Phi_2 = \mathbb{1} \qquad \psi_1 = \mathbb{1} \qquad \psi_2 = \psi. \tag{72}$$

4.3. The Loewner–Einstein–Maxwell connection

A particular stationary generalized Loewner system is obtained by choosing

$$\begin{array}{lll} \Phi_1 = \Phi & \Phi_2 = \mathbb{I} & \Phi_3 = \hat{\varphi} \\ \psi_1 = \mathbb{I} & \psi_2 = \psi & \psi_3 = \hat{\psi}, \end{array} \tag{73}$$

where the (adjoint) eigenfunctions $\hat{\varphi}$ and $\hat{\psi}$ constitute vectors, namely

$$\begin{array}{llll} [\Phi + \psi^\dagger + \hat{\varphi}\hat{\psi}^\dagger, S] = 0 & & & \\ \Phi_y = S\Phi_x & \psi_y = S^\dagger\psi_x & \hat{\varphi}_y = S\hat{\varphi}_x & \hat{\psi}_y = S^\dagger\hat{\psi}_x. \end{array} \tag{74}$$

Comparison with the Einstein–Maxwell field equations (61) shows that we may employ the identifications

$$S = -\frac{\mathrm{i}}{\rho}F \qquad \Phi = H \qquad \hat{\varphi} = 2\varphi \qquad \hat{\psi} = \sigma\varphi. \tag{75}$$

Thus, the Loewner equations $(74)_{2,4,5}$ are satisfied. As in sections 3.2 and 3.3, a canonical definition for ψ is

$$\psi = \sigma H \sigma \tag{76}$$

so that $(74)_3$ is identically satisfied. Indeed, a straightforward calculation analogous to that leading to (40) shows that

$$\psi^\dagger = -H + 2\varphi\varphi^\dagger\sigma + 2(F + \mathrm{i}\nu\mathbb{I}) \tag{77}$$

and hence the remaining commutator relation $(74)_1$ holds. Accordingly, the Einstein–Maxwell equations for axisymmetric stationary space-times may be interpreted as a stationary generalized Loewner system. It is noted that the nonlocal linear representation of the latter given by

$$\phi_y = S\phi_y \qquad \lambda\phi = \Phi\phi + M(\psi, \phi) + \hat{\varphi}M(\hat{\psi}, \phi) \tag{78}$$

where λ is a constant parameter, differs from the standard Lax pairs for the axisymmetric stationary Einstein–Maxwell field equations [18] in that it involves 2×2 rather than 3×3 matrices.

4.4. The Einstein–(Maxwell)^N equations

We conclude our considerations with the remark that, in the same manner, the stationary generalized Loewner system

$$\begin{array}{llll} \left[\Phi + \psi^\dagger + \sum_n \hat{\varphi}_n\hat{\psi}_n^\dagger, S\right] = 0 & & & \\ \Phi_y = S\Phi_x & \psi_y = S^\dagger\psi_x & \hat{\varphi}_{ny} = S\hat{\varphi}_{nx} & \hat{\psi}_{ny} = S^\dagger\hat{\psi}_{nx} \end{array} \tag{79}$$

may be related to

$$\nabla H = \frac{\mathrm{i}}{\rho}F\tilde{\nabla}H \qquad \nabla\varphi_n = \frac{\mathrm{i}}{\rho}F\tilde{\nabla}\varphi_n \tag{80}$$

subject to the constraint

$$\Re(H) = F + \sum_n \Re(\varphi_n\varphi_n^\dagger\sigma + \kappa_n\mathbb{I}), \tag{81}$$

where

$$\nabla\kappa_n = \varphi_n^\dagger\sigma\nabla\varphi_n. \tag{82}$$

This system, in turn, is known to be equivalent to Einstein's equations for axisymmetric stationary space-times coupled with an arbitrary number of $U(1)$ gauge fields. These may be written as [8]

$$
\begin{aligned}
\frac{f}{\rho}\nabla\cdot(\rho\nabla\mathcal{E}) &= \left(\nabla\mathcal{E}+2\sum_m\bar{\Phi}_m\nabla\Phi_m\right)\cdot\nabla\mathcal{E}\\
\frac{f}{\rho}\nabla\cdot(\rho\nabla\Phi_n) &= \left(\nabla\mathcal{E}+2\sum_m\bar{\Phi}_m\nabla\Phi_m\right)\cdot\nabla\Phi_n
\end{aligned}
\tag{83}
$$

with $f = \Re(\mathcal{E}) + \sum_m |\Phi_m|^2$. The Geroch-type transformation for the Loewner system set down in section 3 may now readily be extended to capture generalized Loewner systems and, in particular, the integrable cases of Einstein's field equations discussed in this section.

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